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The nonlinear temporal evolution of a disturbance to a stratified mixing layer

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Using a nonlinear critical layer analysis, Goldstein & Leib (1988) derived a set of nonlinear evolution equations governing the spatial growth of a two-dimensional instability wave on a homogeneous incompressible tanh y mixing layer. In this study, we extend this analysis to the temporal growth of the García model of an incompressible stratified shear layer. We consider the stage of the evolution in which the growth first becomes nonlinear, with the nonlinearity appearing inside the critical layer. The Reynolds number is assumed to be just large enough so that the unsteady, nonlinear and viscous terms all enter at the same order of magnitude inside the critical layer. The equations are solved numerically for the inviscid case.

1. Introduction

If a small nearly neutral two-dimensional disturbance is superimposed on a mixing layer, it is well known that the evolution of the disturbance is linear when the amplitude of the disturbance is sufficiently small. However, if the disturbance is unstable and growing, eventually its amplitude will become large enough that nonlinear effects become important. These nonlinear effects first enter inside the critical layer where the phase velocity of the disturbance equals the mean velocity of the base flow.

Goldstein & Leib (1988) considered the case of a homogeneous incompressible mixing layer. They found that, if the magnitude of the disturbance is assumed to be $O(\epsilon)$, then the evolution remains linear provided $\epsilon^{1/2} \ll \mu$, where μ denotes the order of magnitude of the departure of the wavenumber α of the disturbance from its neutral value α_0 and consequently the timescale (or lengthscale) upon which the disturbance develops. During this linear stage, the amplitude A grows like $a_0 \mathrm{e}^{\mu\sigma t}$, where σ is the (constant) growth rate. Goldstein & Leib found that the evolution first became nonlinear when $\mu \sim O(\epsilon^{1/2})$, and in that regime the amplitude was governed by a set of nonlinear evolution equations (their equations (3.15)–(3.20)). This flow regime had earlier been studied, in the context of forced Rossby waves, by Warn & Warn (1978) and Stewartson (1978). The analysis of Goldstein & Leib was later extended (Goldstein & Hultgren 1988) to the viscous case for large Reynolds numbers, $Re^{-1} = \epsilon^{3/2}\lambda$; these scalings were chosen so that nonlinearity, time dependence and viscosity all enter into the vorticity equation inside the critical layer at the same order. It was found that the critical layer aged into a quasi-equilibrium one, and in the viscous case the initial exponential growth of the instability wave was converted into a weak algebraic growth during the roll-up process. This led to a subsequent stage of evolution, studied in more detail by Hultgren (1992). This next stage involved an equilibrium-type critical layer, analogous to that studied by Benney & Bergeron (1969), but with variable vorticity inside the vortex cores unlike the vorticity-homogenized solution of Benney & Bergeron; by contrast, in the studies of Warn & Warn and Stewartson because of the forcing the vorticity inside the cores did homogenize. Hultgren also compared the theory with experimental results, and found what he considered to be very good agreement. Goldstein and coworkers subsequently extended the analysis to a number of other problems, for example hypersonic boundary layers (Goldstein & Wundrow 1990). For a discussion of previous studies of the homogeneous problem, the reader is referred to Goldstein & Leib (1988), and for a review of nonlinear critical layers in general to Maslowe (1986).

In this study, we also include the effect of stratification by considering the García model of a stratified shear layer (e.g. Drazin & Howard 1966), for which the base stream function $\psi^{(0)}(y)$ and temperature $\theta^{(0)}(y)$ are given by

$$\left.\begin{aligned}\psi^{(0)}(y) &= \log\cosh y \sim y^2/2 - y^4/12 + \ldots,\\ \theta^{(0)}(y) &= \tanh^3 y \sim y^3 - y^5 + \ldots,\end{aligned}\right\} \tag{1.1}$$

where the expansions are valid as $y \to 0$. We shall see in §3 that this leads to a set of equations similar to that in Goldstein & Leib (1988). The García model has previously been studied by Engevik (1982) who considered the Stuart–Watson-type nonlinear stability and by Mallier (1994) who modelled the equilibrium states that have been observed in numerical simulations. In our study, we shall retain viscous terms during the derivation of the evolution equations. However, we shall solve these evolution equations for the inviscid case only, since that will enable us to use a characteristic scheme due to Goldstein & Wundrow (1990).

2. Outer expansion

We consider two dimensional waves of magnitude $O(\epsilon)$ of the form $e^{\pm i\alpha x}$ superimposed on the García model of a stratified shear layer, where $\alpha = \alpha_0 - \epsilon^{1/2}\alpha_1$ is close to the neutral wavenumber α_0. We shall introduce a slow timescale $T = \epsilon^{1/2}t$ and assume the Reynolds number is large, $Re^{-1} = \epsilon^{3/2}\lambda$, where λ is the so-called Benney–Bergeron parameter, so that using the Boussinesq approximation the governing equations become

$$\left.\begin{aligned}\epsilon^{1/2}\nabla^2\psi_T - J(\psi, \nabla^2\psi) + J_0\,\theta_x &= \epsilon^{3/2}\lambda\nabla^4\psi,\\ \epsilon^{1/2}\theta_T - J(\psi,\theta) &= \epsilon^{3/2}\lambda\, Pr^{-1}\nabla^2\theta,\end{aligned}\right\} \tag{2.1}$$

where $J(a,b) = a_x b_y - a_y b_x$ is a Jacobian, J_0 is an overall Richardson number and Pr is the Prandtl number. The outer expansion in the limit as $y \to 0$ will be of the form

$$\psi \sim \psi^{(0)}(y) + \epsilon\psi^{(1)}(x,y,t) + \epsilon^{3/2}\psi^{(3/2)}(x,y,t) + \epsilon^2\psi^{(2)}(x,y,t) + \ldots, \tag{2.2}$$

where

$$\left.\begin{aligned}\psi^{(1)} &= \psi_0^{(1)}(y,T) + (\psi_1^{(1)}(y,T)\,e^{i\alpha x} + \text{c.c.}),\\ \psi^{(3/2)} &= \psi_0^{(3/2)}(\theta,T) + \sum_{m=1}^{\infty}(\psi_m^{(3/2)}(y,T)\,e^{im\alpha x} + \text{c.c.}),\end{aligned}\right\} \tag{2.3}$$

with a similar expression for θ, where $\theta^{(1)}$, $\theta^{(3/2)}$, etc. are given by similar expressions. In general we shall use the notation $\psi_m^{(l)}$ to mean the term at $O(\epsilon^l)$ accompanying $e^{i\alpha m x}$, and we shall introduce the operator

$$\mathscr{L}_m\phi = (\partial_{yy}^2 + (\alpha_0(\alpha_0+1)\operatorname{sech}^2 y - m^2\alpha_0^2))\,\phi. \tag{2.4}$$

Substituting this expansion into the governing equations yields an hierarchy of equations. As noted in §1, these scalings are chosen so that nonlinearity, time dependence and viscosity all enter into the vorticity equation (3.1) at the same order inside the critical layer.

At $O(\epsilon)$, from linear stability theory (Drazin & Howard 1966), we know that $J_0 = (\alpha_0 - 1)(\alpha_0 + 2)/3$ and

$$\psi_1^{(1)} = A(T)\,\mathrm{sech}^{\alpha_0} y, \quad \theta_1^{(1)} = 3A(T)\,\mathrm{sech}^{3+\alpha_0} y \sinh y. \tag{2.5}$$

We also find that the leading-order change in the mean flow due to diffusive effects is given by

$$\psi_0^{(1)} = \lambda T \,\mathrm{sech}^2 y, \quad \theta_0^{(1)} = \frac{3\lambda T}{2Pr}\,\mathrm{sech}^5 y (7\sinh y - \sinh 3y). \tag{2.6}$$

At $O(\epsilon^{3/2})$, we find that $\psi_0^{(3/2)} = \theta_0^{(3/2)} = 0$ and

$$\mathscr{L}_1 \psi_1^{(3/2)} = -2\alpha_0 \alpha_1 A \,\mathrm{sech}^{\alpha_0} y - \frac{2\mathrm{i}(\alpha_0^2 + \alpha_0 - 1)A'}{\alpha_0 \sinh y \cosh^{1+\alpha_0} y}, \tag{2.7a}$$

$$\psi_1^{(3/2)} \sim C_1^{(3/2)} - \alpha_1 A + y(D_1^{(3/2)(\pm)} + 2\mathrm{i}A'(\alpha_0 + 1 - \alpha_0^{-1})(1 - \log|y|)) + \ldots, \tag{2.7b}$$

$$\theta_1^{(3/2)} \sim 3\mathrm{i}A'\alpha_0^{-1} + y(3C_1^{(3/2)} - 3\alpha_1 A) + \ldots, \tag{2.7c}$$

where this expansion is valid as $y \to 0$, and $C_1^{(3/2)}(T)$ and $D_1^{(3/2)(\pm)}(T)$ are (as yet) undetermined functions of the slow timescale T and the superscripts '+' and '−' refer to the solutions above and below the critical layer respectively.

Multiplying the equation for $\mathscr{L}_1 \psi_1^{(3/2)}$ by $\psi_1^{(1)}$ and integrating between $-\infty$ and ∞, taking the Cauchy principal value, and enforcing the homogeneous boundary condition on $\psi_1^{(3/2)}$, we obtain a solvability condition,

$$D_1^{(3/2)(+)} - D_1^{(3/2)(-)} = 2\alpha_1 \alpha_0 B(\tfrac{1}{2}, \alpha_0) A, \tag{2.8}$$

where $B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,\mathrm{d}t$ is the beta function. The jump in $\psi_{1y}^{(3/2)}$ across the critical layer is given by

$$[\psi_{1y}^{(3/2)}]_{0-}^{0+} = D_1^{(3/2)(+)} - D_1^{(3/2)(-)} = 2\alpha_1 \alpha_0 B(\tfrac{1}{2}, \alpha_0) A. \tag{2.9}$$

In the early stages of the disturbance (the so-called linear-growth critical layer), we know that the jump across the critical layer is simply

$$D_1^{(3/2)(+)} - D_1^{(3/2)(-)} = 2\pi A'(\alpha_0 + 1 - \alpha_0^{-1}), \tag{2.10}$$

since during this stage the logarithmic phase shift is $-\mathrm{i}\pi$; thus, in the early stages of the disturbance, the amplitude $A(T)$ obeys $A' = \sigma A$, where σ is the growth rate,

$$\sigma = \frac{\alpha_0^2 \alpha_1 B(\frac{1}{2}, \alpha_0)}{\pi(\alpha_0^2 + \alpha_0 - 1)}. \tag{2.11}$$

This gives us an initial condition, $A \to a_0 \mathrm{e}^{\sigma T}$ as $T \to -\infty$.

For the higher harmonics, we find that for $m \geqslant 2$,

$$\psi_m^{(3/2)} \sim C_m^{(3/2)}(T) + O(y), \quad \theta_m^{(3/2)} \sim 3C_m^{(3/2)}(T)y + O(y^2). \tag{2.12}$$

3. Inner expansion

As in Goldstein & Leib (1988), near the critical layer at $y = 0$, we shall introduce the rescaled variables $Y = \epsilon^{-1/2}y$, $\xi = \alpha x$, $\Psi = \epsilon\psi$ and $\Theta = \epsilon^{-3/2}\theta$, which leads to the governing equations in the critical layer,

$$\Psi_{YYT} + \epsilon\alpha^2\Psi_{\xi\xi T} - \alpha J_{CL}(\Psi, \Psi_{YY} + \epsilon\alpha^2\Psi_{\xi\xi}) + \epsilon J_0 \alpha \Theta_\xi - \lambda(\Psi_{YYYY} + 2\epsilon\alpha^2\Psi_{\xi\xi YY} + \epsilon^2\alpha^4\Psi_{\xi\xi\xi\xi}) = 0, \tag{3.1a}$$

$$\Theta_T - \alpha J_{CL}(\Psi, \Theta) - \lambda Pr^{-1}(\Theta_{YY} + \epsilon\alpha^2\Theta_{\xi\xi}) = 0, \tag{3.1b}$$

where $J_{CL}(a,b) = a_\xi b_Y - a_Y b_\xi$.

The form of the outer solution written in inner variables suggests that

$$\Psi \sim \Psi^{(0)} + \epsilon^{1/2}\Psi^{(1/2)} + (\epsilon \log \epsilon^{1/2})\,\Psi^{(1l)} + \epsilon\Psi^{(1)} + \ldots, \tag{3.2}$$

with a similar expression for Θ. Substituting this expansion into the governing equations yields an hierarchy of equations.

At $O(\epsilon^0)$, from the outer solution, we know that

$$\Psi^{(0)} = Y^2/2+\lambda T+A(T)\,e^{i\xi}+A^*(T)\,e^{-i\xi} \tag{3.3}$$

and the behaviour of $\Theta^{(0)}$ as $Y\to\pm\infty$ leads us to write

$$\Theta^{(0)} = Y^3+6\lambda\,Pr^{-1}TY+(3e^{i\xi}(AY+iA'\alpha_0^{-1})+\text{c.c.})+\vartheta^{(0)}, \tag{3.4}$$

which leads to the following equation for $\vartheta^{(0)}$:

$$\begin{aligned}(\partial_T+\alpha_0 Y\partial_\xi-\alpha_0 i(A\,e^{i\xi}-A^*\,e^{-i\xi})\,\partial_Y-\lambda\,Pr^{-1}\,\partial^2_{YY})\,\vartheta^{(0)}\\ = 6\lambda\,Pr^{-1}\,T\alpha_0 i(A\,e^{i\xi}-A^*\,e^{-i\xi})-3i\alpha_0^{-1}(A''\,e^{i\xi}-A^{*''}\,e^{-i}\xi)+3\alpha_0 i(A^2\,e^{2i\xi}-A^{*2}\,e^{-2i\xi})\end{aligned} \tag{3.5}$$

together with the boundary condition that $\vartheta^{(0)}\to 0$ as $Y\to\pm\infty$.

At $O(\epsilon^{1/2})$, we find that the solution which matches to the outer expansion is

$$\Psi^{(1/2)} = (C_1^{(3/2)}+\alpha_1 A)\,e^{i\alpha x}+\text{c.c.}+\sum_{m-2}^{\infty}(C_m^{(3/2)}\,e^{im\alpha x}+\text{c.c.}). \tag{3.6}$$

At $O(\epsilon^1)$, the behaviour of $\Phi^{(1)}_{YY}$ as $Y\to\pm\infty$ leads us to write

$$\Psi^{(1)}_{YY} = -Y^2-2\lambda T-\alpha_0(A\,e^{i\xi}+A^*\,e^{-i\xi})+Q^{(1)}, \tag{3.7}$$

which leads to the following equation for $Q^{(1)}$:

$$\begin{aligned}(\partial_T+\alpha_0 Y\partial_\xi-\alpha_0 i(A\,e^{i\xi}-A^*\,e^{-i\xi})\,\partial_Y-\lambda\partial^2_{YY})\,Q^{(1)}\\ = 2(\alpha_0^2+\alpha_0-1)(A'\,e^{i\xi}+A^{*\prime}\,e^{-i\xi})-\tfrac{1}{3}\alpha_0(\alpha_0-1)(2+\alpha_0)\,\vartheta^{(0)}_\xi\end{aligned} \tag{3.8}$$

together with the boundary condition that $Q^{(1)}\to 0$ as $Y\to\pm\infty$. We also have a jump condition across the critical layer:

$$\frac{1}{2\pi}\int_0^{2\pi}\int_{-\infty}^{\infty} e^{-i\xi}\,Q^{(1)}\,dY\,d\xi = [\psi^{(3/2)}_{1y}]^{0+}_{0-} = 2\alpha_1\alpha_0\,B(\tfrac{1}{2},\alpha_0)\,A. \tag{3.9}$$

Thus we have the coupled nonlinear evolution equations (3.5), (3.8) together with the jump condition (3.9) and the boundary conditions that $\vartheta^{(0)}\to 0$ and $Q^{(1)}\to 0$ as $Y\to\pm\infty$ and the initial condition that $A\to a_0 e^{\sigma T}$ as $T\to-\infty$. This initial condition for $A(T)$ together with (3.5), (3.8) tells us that as $T\to-\infty$,

$$Q^{(1)}\to a_0\,e^{i\xi+\sigma T}(E(z)\,T+F(z))+\text{c.c.},\quad \vartheta^{(0)}\to a_0\,e^{i\xi+\sigma T}(G(\tilde z)\,T+H(\tilde z))+\text{c.c.}, \tag{3.10}$$

where $z=(i\alpha_0\lambda^{-1})^{1/3}\,(Y-(i\sigma/\alpha_0))$ and $\tilde z=Pr^{1/3}z$, and E, F, G and H satisfy the inhomogeneous Airy's equations

$$E_{zz}-zE=\left(\frac{\lambda}{i\alpha_0}\right)^{2/3}\left(\frac{i\alpha_0(\alpha_0-1)(2+\alpha_0)}{3\lambda}\right)G(Pr^{1/3}z), \tag{3.11a}$$

$$F_{zz}-zF=\frac{1}{\lambda}(\lambda i\alpha_0)^{2/3}E(z)+\left(\frac{\lambda}{i\alpha_0}\right)^{2/3}\left(\frac{i\alpha_0(\alpha_0-1)(2+\alpha_0)}{3\lambda}\right)H(Pr^{1/3}z)+\frac{2\sigma}{\lambda}\left(\frac{\lambda}{i\alpha_0}\right)^{2/3}(1-\alpha_0-\alpha_0^2), \tag{3.11b}$$

$$G_{\tilde z\tilde z}-\tilde zG=-6i\alpha_0\left(\frac{\lambda Pr^{-1}}{i\alpha_0}\right)^{2/3}, \tag{3.11c}$$

$$H_{\tilde z\tilde z}-\tilde zH=\frac{Pr}{\lambda}\left(\frac{\lambda Pr^{-1}}{i\alpha_0}\right)^{2/3}G(\tilde z)+\frac{3i\,\mathrm{Pr}\sigma^2}{\alpha_0\lambda}\left(\frac{\lambda Pr^{-1}}{i\alpha_0}\right)^{2/3}, \tag{3.11d}$$

subject to the integral constraints (arising from the jump conditions across the critical layer) that

$$\int_{-\infty}^{\infty} E(z)\,\mathrm{d}z = \int_{-\infty}^{\infty} G(\tilde{z})\,\mathrm{d}\tilde{z} = \int_{-\infty}^{\infty} H(\tilde{z})\,\mathrm{d}\tilde{z} = 0, \tag{3.12a}$$

$$\int_{-\infty}^{\infty} F(z)\,\mathrm{d}z = 2\alpha_1\alpha_0\left(\frac{\lambda}{\mathrm{i}\alpha_0}\right)^{1/3} B(\tfrac{1}{2},\alpha_0). \tag{3.12b}$$

When $\alpha_0 = 1$ (so that $J_0 = 0$, corresponding to the unstratified case), the equation for $\vartheta^{(0)}$ is decoupled and we are left with the system of equations for a passive scalar in a homogeneous $\tanh y$ mixing layer, namely

$$(\partial_T + Y\partial_\xi - \mathrm{i}(A\,\mathrm{e}^{\mathrm{i}\xi} - A^*\,\mathrm{e}^{-\mathrm{i}\xi})\,\partial_Y - \lambda\partial^2_{YY})\,Q^{(1)} = 2(A'\,\mathrm{e}^{\mathrm{i}\xi} + A^{*\prime}\,\mathrm{e}^{-\mathrm{i}\xi}), \tag{3.13}$$

and

$$\begin{aligned}(\partial_T + Y\partial_\xi - \mathrm{i}(A\,\mathrm{e}^{\mathrm{i}\xi} - A^*\,\mathrm{e}^{-\mathrm{i}\xi})\,\partial_Y - \lambda\,Pr^{-1}\,\partial^2_{YY})\,\vartheta^{(0)} \\ = 6\lambda\,Pr^{-1}T\mathrm{i}(A\,\mathrm{e}^{\mathrm{i}\xi} - A^*\,\mathrm{e}^{-\mathrm{i}\xi}) - 3\mathrm{i}(A''\,\mathrm{e}^{\mathrm{i}\xi} - A^{*\prime\prime}\,\mathrm{e}^{-\mathrm{i}\xi}) + 3\mathrm{i}(A^2\,\mathrm{e}^{2\mathrm{i}\xi} - A^{*2}\,\mathrm{e}^{-2\mathrm{i}\xi})\end{aligned} \tag{3.14}$$

together with the jump condition

$$\frac{1}{2\pi}\int_0^{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\xi}\,Q^{(1)}\,\mathrm{d}Y\,\mathrm{d}\xi = 4\alpha_1 A. \tag{3.15}$$

4. Numerical computation

The coupled nonlinear evolution equations (3.5), (3.8), (3.9) must be solved numerically. For the inviscid case ($\lambda = 0$), this is comparatively straightforward to accomplish using a characteristic scheme similar to that described in Goldstein & Wundrow (1990) for a similar problem. For the inviscid case, we can see that $A \to a_0\,\mathrm{e}^{\sigma T}$ as $T \to -\infty$ (where a_0 is a constant which we can choose to be 1 by shifting T), and

$$\tilde{Q}^{(1)} \to -\frac{2\mathrm{i}(\alpha_0^2 + \alpha_0 - 1)\,A'(T)\,\mathrm{e}^{\mathrm{i}\xi}}{\alpha_0(Y - \mathrm{i}\sigma/\alpha_0)} + \text{c.c.}, \tag{4.1a}$$

$$\vartheta^{(0)} \to -\frac{3A''(T)\,\mathrm{e}^{\mathrm{i}\xi}}{\alpha_0^2(Y - \mathrm{i}\sigma/\alpha_0)} + \text{c.c.}, \tag{4.1b}$$

where $\tilde{Q}^{(1)} = Q^{(1)} - \frac{1}{3}(\alpha_0 - 1)(\alpha_0 + 2)\,\vartheta^{(0)}_Y$. One then introduces Lagrangian (characteristic) coordinates ξ_0, Y_0 and $\tau = T$ via

$$\frac{\mathrm{d}\xi}{\mathrm{d}\tau} = \alpha_0 Y \quad\text{and}\quad \frac{\mathrm{d}Y}{\mathrm{d}\tau} = 2\alpha_0 A\sin\xi, \tag{4.2}$$

where the determinant $\partial(\xi, Y)/\partial(\xi_0, Y_0) = 1$ and $\xi_0 \equiv \xi|_{\tau=\tau_0}$ and $Y_0 \equiv Y|_{\tau=\tau_0}$ as $\tau_0 \to -\infty$. With this change of variables, we can write (3.5), (3.8) in the form

$$\frac{\mathrm{d}\vartheta^{(0)}}{\mathrm{d}\tau} = \frac{6A''\sin\xi}{\alpha_0} - 6\alpha_0 A^2\sin 2\xi, \quad \frac{\mathrm{d}\tilde{Q}^{(1)}}{\mathrm{d}\tau} = 4(\alpha_0^2 + \alpha_0 - 1)\,A'\cos\xi, \tag{4.3}$$

where we have used the fact that $H_Y \equiv \partial(\xi, H)/\partial(\xi_0, Y_0)$. These equations have a solution

$$\vartheta^{(0)} = -Y^3 + 6A'\sin\xi/\alpha_0 - 6AY\cos\xi + F(\xi_0, Y_0), \tag{4.4a}$$

$$\tilde{Q}^{(1)} = 4(\alpha_0^2 + \alpha_0 - 1)\,A\cos\xi + (\alpha_0^2 + \alpha_0 - 1)\,Y^2 + G(\xi_0, Y_0), \tag{4.4b}$$

where F and G depend only on ξ_0 and Y_0 (and not on τ). The solutions to (4.2) and (4.3) have the following symmetry;

$$\xi(\xi_0, Y_0, \tau) = \xi(-\xi_0, -Y_0, \tau), \quad Y(\xi_0, Y_0 \tau) = -Y(-\xi_0, -Y_0, \tau), \qquad (4.5a, b)$$

$$\tilde{Q}^{(1)}(\xi_0, Y_0, \tau) = \tilde{Q}^{(1)}(-\xi_0, -Y_0, \tau), \quad \vartheta^{(0)}(\xi_0, Y_0, \tau) = -\vartheta^{(0)}(-\xi_0, -Y_0, \tau), \quad (4.5c, d)$$

and consequently calculations need only be done for $0 \leqslant Y_0 < \infty$. Since

$$\textstyle\int_0^{2\pi} \int_{-\infty}^{\infty} \vartheta^{(0)} \cos \xi \, \mathrm{d} Y \mathrm{d}\xi = 0,$$

we can write the jump condition (3.9) as

$$\frac{1}{\pi} \int_0^{2\pi} \int_0^{\infty} \tilde{Q}^{(1)} \cos \xi \, \mathrm{d} Y_0 \, \mathrm{d}\xi_0 = 2\alpha_1 \alpha_0 B(\tfrac{1}{2}, \alpha_0) A. \qquad (4.6)$$

We know that $\mathrm{d}\Theta^{(0)}/\mathrm{d}\tau = 0$, which enables us to generate asymptotic expansions for Y and subsequently ξ and $\tilde{Q}^{(1)}$,

$$Y \sim Y_0 + \frac{2(A(\tau_0) \cos \xi_0 - A(\tau) \cos \xi)}{Y_0} + \frac{2(A'(\tau) \sin \xi_0 - A'(\tau_0) \sin \xi)}{Y_0^2} + O\left(\frac{1}{Y_0^3}\right), \quad (4.7a)$$

$$\xi \sim \xi_0 + \alpha_0 (\tau - \tau_0) Y_0 + O(1/Y_0^2), \qquad (4.7b)$$

$$\tilde{Q}^{(1)} \sim \frac{4(\alpha_0^2 + \alpha_0 - 1) A'(\tau) \sin \xi}{\alpha_0 Y_0} + \frac{4(\alpha_0^2 + \alpha_0 - 1) A''(\tau) \cos \xi}{\alpha_0^2 Y_0^2} + O\left(\frac{1}{Y_0^3}\right), \qquad (4.7c)$$

and consequently we can approximate (4.6) by

$$\frac{1}{\pi} \int_0^{2\pi} \int_0^{M} \tilde{Q}^{(1)} \cos \xi \, \mathrm{d} Y_0 \, \mathrm{d}\xi_0 = 2\alpha_1 \alpha_0 B(\tfrac{1}{2}, \alpha_0) A - \frac{4(\alpha_0^2 + \alpha_0 - 1) A''(\tau)}{\alpha_0^2 M}. \qquad (4.8)$$

The equations (4.2), (4.3) and (4.8) were solved numerically using a fourth-order Runge–Kutta scheme to march forward in time and Simpson's rule to evaluate the integrals in (4.8); the integration was started in the linear regime, using (4.1) to provide initial conditions for $\tilde{Q}^{(1)}$ and $\vartheta^{(0)}$.

5. Numerical results

In figure 1, we plot the evolution of the amplitude A for several values of the Richardson number J_0. We see that for all the cases studied, the growth is initially *linear*, and that when nonlinear terms become important, the solution oscillates about a quasi-equilibrium state. This quasi-equilibrium state is consistent with results from numerical simulations of two-dimensional stratified shear layers, and the oscillation about this state is believed to be related to the phenomenon of vortex nutation observation in numerical simulations. While the Stuart–Watson type theory studied by Engevik (1982) had allowed for an equilibrium state, oscillations about this state were not possible. The effect of stratification upon the evolution of the amplitude appears to be two-fold, with the growth rate during the linear phase and the amplitude of the quasi-equilibrium state both decreasing as J_0 is increased.

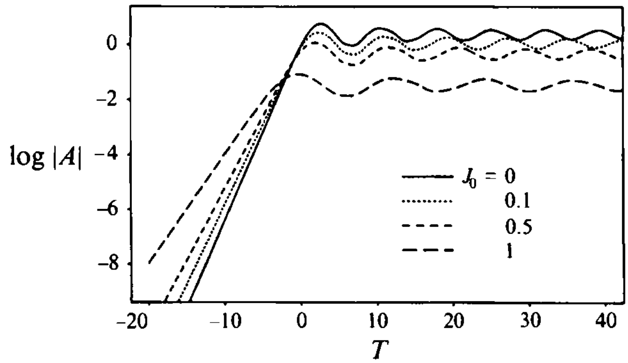

FIGURE 1. Evolution of the amplitude A.

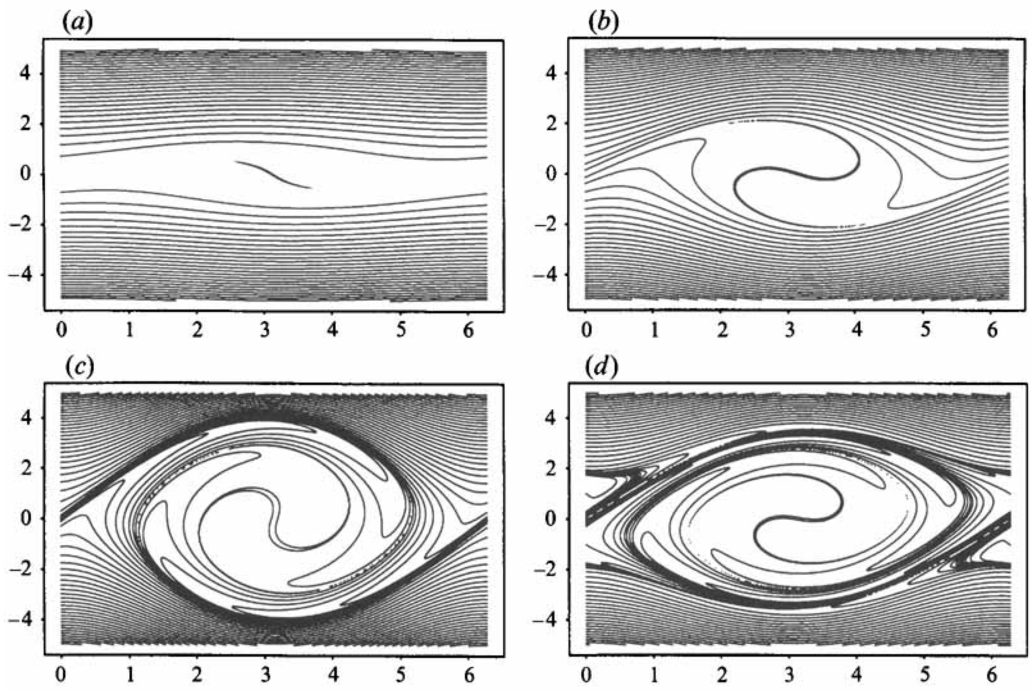

FIGURE 2. Vorticity contours for $J_0 = 0$: (*a*) $t = -2.5$, (*b*) $t = 0$, (*c*) $t = 2.5$, (*d*) $t = 5$.

In figures 2 and 3, we plot the evolution of the vorticity contours for both the unstratified case and a case with fairly strong stratification ($J_0 = 1$), and we see that initially small disturbances grow and wrap up, with sharp gradients developing at the edges of the cat's eyes in the later stages. In figures 4 and 5, we plot the vorticity on a vertical section through the core (on the plane $\xi = \pi$) for the plots shown in figures 2 and 3 respectively. For all the cases shown, and others at different values of J_0 not presented here, we get strong spikes of vorticity developing at the edges of the cat's eyes in the later stages; these correspond to the sharp gradients which appear in the contour plots and which have also been observed in numerical simulations. These spikes would to some extent be smoothed out if viscosity were present. One noticeable effect of stratification is the appearance of a relative trough of vorticity inside the core which other plots not presented here suggest becomes deeper as the Richardson number is

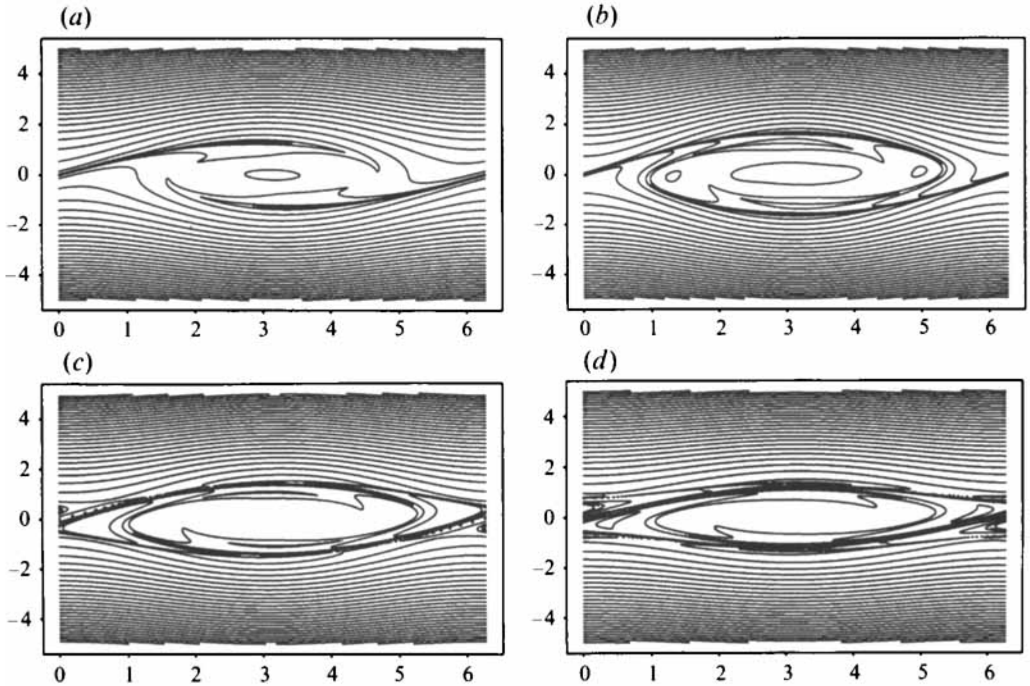

FIGURE 3. As in figure 2 but $J_0 = 1$.

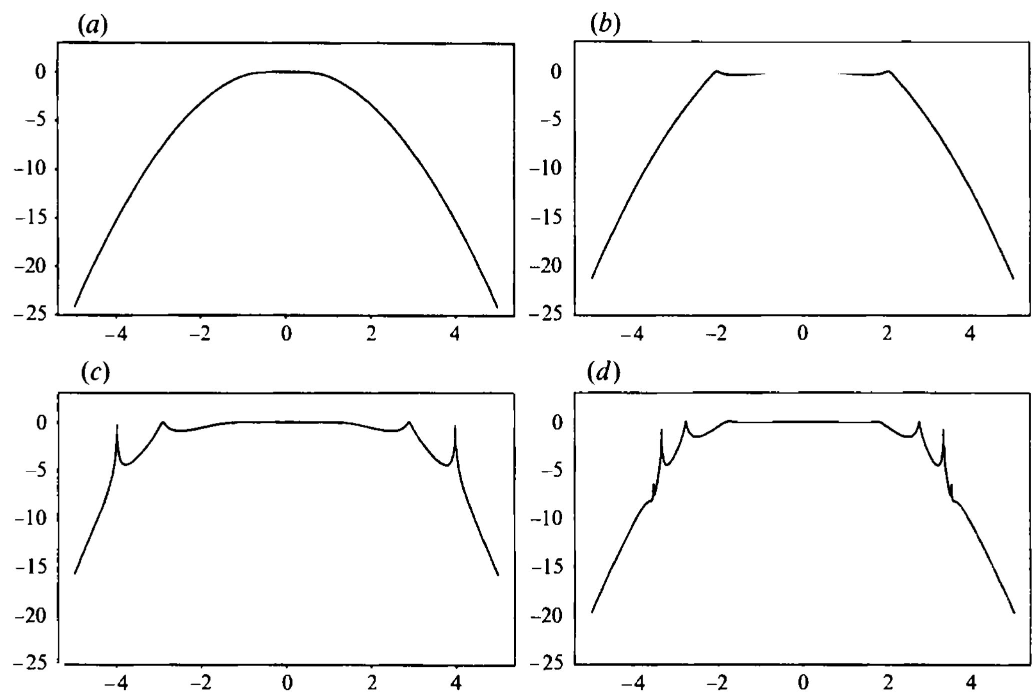

FIGURE 4. Vorticity on line $\xi = \pi$ for $J_0 = 0$: (*a*) $t = -2.5$, (*b*) $t = 0$, (*c*) $t = 2.5$, (*d*) $t = 5$

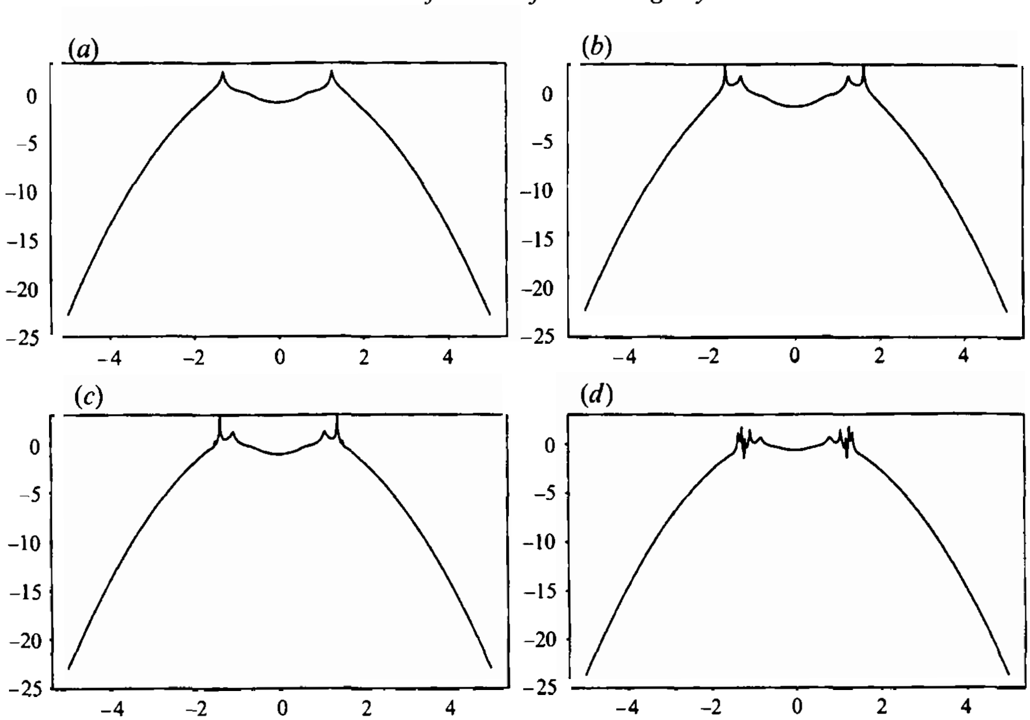

FIGURE 5. As in figure 4 but $J_0 = 1$

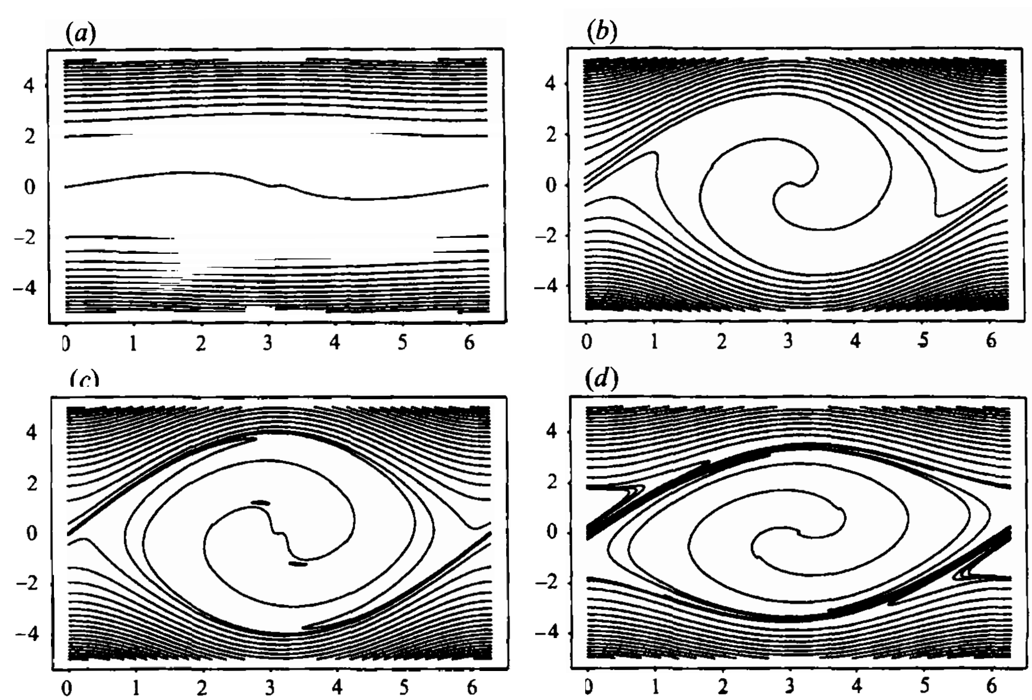

FIGURE 6. Temperature contours for $J_0 = 0$: (a) $= -2.5$, (b) $t = 0$, (c) $t = 2.5$, (d) $t = 5$.

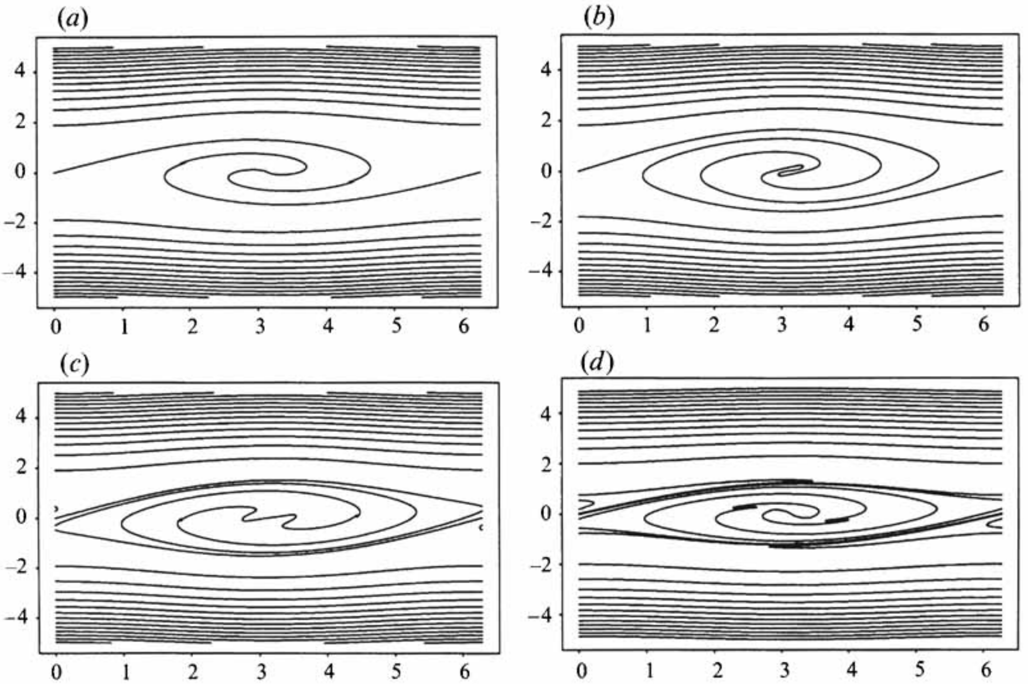

FIGURE 7. As in figure 6 but $J_0 = 1$

increased. Another effect is that the vortex becomes flatter, that is the aspect rat ɔ becomes more extreme, with increasing J_0. Finally in figures 6 and 7, we plot the temperature contours at different times for several different Richardson numbers. Examination of the structure of the temperature on a vertical section through the core indicates that, unlike the vorticity, the temperature inside the core homogenizes fairly rapidly.

6. Concluding remarks

By considering the flow inside the critical layer, we have derived a system of equations (3.5), (3.8), (3.9) governing the temporal evolution of an initially linear disturbance in a stratified incompressible shear layer; this set of equations, together with the numerical solutions for the inviscid case, is the final result of this paper. Comparison with the unstratified case (3.13)–(3.15) suggests that whereas the equation for $\vartheta^{(0)}$ is relatively unchanged from the equation for a passive scalar, the equation for $Q^{(1)}$ has an additional term on the right-hand side which represents the coupling of the temperature to the vorticity equation.

Numerical solutions of these equations for the inviscid case show that both the temperature and vorticity develop sharp gradients at the edge of the core, together with the somewhat surprising finding that a relative trough of vorticity develops inside the core for non-zero Richardson numbers, with this trough presumably being due to the coupling term in the equation for $Q^{(1)}$ mentioned above. By contrast, for the zero-Richardson-number case the vorticity on a section through the core (figure 4) appears to have a slight peak inside the core. The relative trough for non-zero J_0 is slightly surprising because earlier studies of equilibrium states in stratified fluids had predicted either a constant vorticity distribution inside the core (e.g. Maslowe 1972), with this prediction based on the Prandtl–Batchelor theorem, or alternatively a smooth distribution of vorticity inside the core resembling that of the Stuart vortex (e.g. Mallier 1994), with this prediction based on an argument due to Rhines & Young (1983) that homogenization of vorticity inside closed streamlines takes place on two distinct timescales.

For the homogeneous mixing layer, Goldstein & Hultgren (1988) found that one effect of viscosity was that the vorticity distribution was diffused into a more regular pattern than had been the case in the absence of vorticity, and it is reasonable to suppose that a similar situation would hold here if diffusive effects were retained in the numerical computations. It is unclear how the addition of viscosity to the problem would affect the relative trough of vorticity discussed above, and this is an issue that would best be tackled by direct numerical simulation: indeed, it seems a little surprising that none of the numerous simulations of stratified flow to date appear to have examined in detail the structure of the vorticity inside the core, other than simply presenting contour plots which do not resolve the issue raised here, preferring instead to focus on other issues.

Finally, we touch on the issue of using a different model of a stratified shear layer, for example the Hølmboe model for which the base temperature is given by $\theta^{(0)} = \tanh y$. Although it is unclear exactly how the present results would be changed, numerical simulations of the full Navier–Stokes equations (L. P. Wang & M. R. Maxey, 1991, private communication) suggest that, although the gradients that appear at the edge of the core are even more severe for the Hølmboe model than for the García model, the gross features of the flow do not change that much, at least for Richardson numbers of about 0.1. One thing that is clear, however, is that the analysis for the Hølmboe model would be considerably more difficult since the singularity at the critical layer is far worse for that model.

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